Challenge Problems from HKCEE Quick Drill text Book

Part 1: Trigonometry

1. Prove that: $\tan(45° + x) - \tan(45° - x) = 2\tan 2x$
2. Find the general solution of the equation: $\sin 9x \sin 8x - \sin 7x \sin 6x = 0$
3. Prove that: $\dfrac{\cos x - 5\cos 5x + \cos 9x}{\sin x - 5\sin 5x + \sin 9x} = \cot 5x$
4. Using the formula: $\cos 3x = \cos(x + 2x)$, prove that: $\cos 3x = 4\cos^3 x - 3\cos x$
    a. What is the value of $\cos x$ if $\cos 3x = 1$
5. Show that $\sin 3x = 3\sin x - 4\sin^3 x$
    a. Using the formula above, Prove that $\sin 18° = \dfrac{\sqrt{5} - 1}{4}$
6. Given that: $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$, show that: $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$
    a. Using the formula above, solve $\tan^3 x - 3\tan^2 x - 3\tan x + 1 = 0$
7. Prove that: $\cot(x + y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$
    a. Show that if $A + B + C = \pi$, where $0 < A, B, C < \pi$ then:
    $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$
    b. Show that if $w + x + y + z = \pi$ and $0 < w, x, y, z < \pi$, then

$$\cot w \cot x \cot y + \cot w \cot x \cot z + \cot x \cot y \cot z + \cot w \cot y \cot z = \cot w + \cot x + \cot y + \cot z$$

Part 2: Algebra:

Property of Root Functions:

$$\left(\sqrt{a} - \sqrt{b}\right)^2 = \left(\sqrt{a} - \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = a + b - 2\sqrt{ab}$$

Therefore: $\sqrt{a + b - 2\sqrt{ab}} = \sqrt{a} - \sqrt{b}$

Q: Given that $\sqrt{3} \approx 1.73205080$, evaluate the value of $\sqrt{7 - 4\sqrt{3}}$ accurate to 5 decimal places without using a calculator

Part 3: Sequences: p83

Prove the following sequences algebraically:

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$1^4 + 2^4 + 3^4 + ... + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

Prove that: $\frac{1}{1\times 2\times 3} + \frac{1}{2\times 3\times 4} + ... + \frac{1}{n(n+1)(n+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$ for any positive integer "n"

Find the value(s) of "n" such that: $\frac{1}{1\times 2\times 3} + \frac{1}{2\times 3\times 4} + ... + \frac{1}{n(n+1)(n+2)} < \frac{2}{9}$

Prove that: $2\times 2^1 + 3\times 2^2 + ...(n+1)2^n = n\times 2^{n+1}$

Prove that: $1\times 3 + 2\times 5 + 3\times 7 + ... + n(2n+1) = \frac{n(n+1)(4n+5)}{6}$